Detecting Binary Black Holes With Efficient and Reliable Templates

T. Damour[†], B.R. Iyer[‡] and B.S. Sathyaprakash[§*]

[†] *IHES, Bur-Sur Yvette, France.*

[‡] *Raman Research Institute, Bangalore, India.*

[§] *Department of Physics and Astronomy, Cardiff University, Cardiff, UK.*

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Abstract

Detecting binary black holes in interferometer data requires an accurate knowledge of the orbital phase evolution of the system. From the point of view of data analysis one also needs fast algorithms to compute the templates that will employed in searching for black hole binaries. Recently, there has been progress on both these fronts: On the one hand, re-summation techniques have made it possible to accelerate the convergence of poorly convergent asymptotic post-Newtonian series and derive waveforms beyond the conventional adiabatic approximation. We now have a waveform model that extends beyond the inspiral regime into the plunge phase followed by the quasi-normal mode ringing. On the other hand, explicit Fourier domain waveforms have been derived that make the generation of waveforms fast enough so as not to be a burden on the computational resources required in filtering the detector data. These new developments should make it possible to efficiently and reliably search for black hole binaries in data from first interferometers.

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*B.Sathyaprakash@CARDIFF.AC.UK

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1 Introduction

One of the theoretical challenges faced in the analysis of data from gravitational wave (GW) detectors is the construction of an efficient and reliable set of templates to search for binary black holes (BH) and neutron stars (NS). Firstly, a fast algorithm is needed to compute the waveforms as the parameter space of compact binaries is quite large requiring several 100,000 templates to cover an interesting range of systems, making it prohibitively expensive to store the templates digitally. Secondly, the templates we employ in our search must be sufficiently accurate representations of the true GW signal from a BH binary so that we miss out, as a result of the mis-match, only a small fraction of all possible events. In this talk we will report and discuss some recent progress made on both these fronts: We now have algorithms to compute templates, both in the time- and frequency-domains, that take only one-half of the computational time to filter out the detector data through these templates. In the time-domain this is achieved by solving a pair of ordinary differential equations (ODEs) [1], rather than using parametric representations of the phasing formulas that require solving integrals of rational polynomials [2]. In the frequency-domain we have derived explicit analytic expressions of the Fourier components [3] that constitute an accurate representation of the time-domain signal truncated at, or slightly prior to, the last stable orbit (LSO). In addition, to an explicit Fourier domain phasing formula we also have a pair of ODEs in the frequency-domain giving the Fourier phase as a function of frequency [1].

In the test mass approximation, the LSO occurs at a frequency $f_{\rm LSO} = 220(20M_\odot/M)$ Hz, while for comparable masses it is most likely to occur at higher frequencies (see, for example Ref. [2]). Until recently, time-domain template waveforms were truncated at the LSO since we were totally unaware of how to continue the waveform beyond the last stable orbit. This time truncation, harmless as it may sound, does mean a specific modelling of the frequency content of the merger signal which may or may not be detrimental to extracting the true GW signal, depending on how the true signal behaves. Our experiments showed that if the merger signal terminates quickly, say over less than half an orbital time-scale, then there is no harm in using the time-truncated waveforms, or their Fourier-domain counterparts [3], as search templates. However, recent theoretical progress in the Hamiltonian description of [4], and the effective one-body approach to solving the binary black hole dynamics has shed some light on how a binary may end its life after crossing the LSO [5]. More precisely, one has now a reasonably good picture of the last three milli-seconds in the life of a stellar mass static black hole binary when two Schwarzschild black holes merge leading to a single spinning Kerr hole. It has been suggested that a good strategy to detect compact binaries is to use as search templates the waveforms based on the effective one-body approach. These waveforms comprise the inspiral and merger phases followed by the quasi-normal mode ringing of the newly formed spinning black hole. Such a template bank should be augmented by an extended set of search templates, that result from a slight variation of the effective one-body waveforms, so as to take into account the uncertainty that lies in the modelling of the late stages in the evolution of these systems.

A combination of the sensitivity of initial interferometers and the expected compact binary merger rates in the Universe has meant that the first likely binary source, an interferometer network is likely to observe, is the merger of stellar mass black holes [6, 3, 1]. The LIGO-VIRGO-GEO interferometer network has its peak sensitivity in the frequency interval 150-400 Hz, which corresponds to a range 11-30 $M_\odot$ of the total mass of a system, whose

LSO, and hence the most dominant part of the inspiral signal, occurs in the least noisy band-width of the network. Thus, these are our best candidate sources. Fortunately, the number of templates needed to search for binaries in this range of masses is less than 10. Hence we will not increase the computational cost, nor will we worsen the statistics (false alarm/dismissal rates) by enhancing the number of templates in this range of masses.

2 Post-Newtonian waveform

The post-Newtonian (PN) equations of motion and wave generation formalisms, when applied to a compact binary evolution, facilitate the computation of the two polarizations h_+ and $h_\times$ of the gravitational wave emitted by the system. For a system in circular orbit, located at a distance r from the Earth, the two polarisations are given as post-Newtonian expansions in the invariant velocity v by the following expressions[7, 8]

$$h_{+,\times} = \frac{2M\eta v^2}{r}\left[H^{(0)}_{+,\times} + \frac{\delta M}{M}H^{(1)}_{+,\times}v + H^{(2)}_{+,\times}v^2 + \frac{\delta M}{M}H^{(3)}_{+,\times}v^3 + H^{(4)}_{+,\times}v^4\right], \tag{1}$$

where

$$H^{(0)}_{+} = -(1+\cos^2 i)\cos 2\varphi, \qquad H^{(0)}_{\times} = -2\cos i \sin 2\varphi, \qquad \ldots \tag{2}$$

$M = m_1 + m_2$ and $\eta = m_1 m_2/M^2$ are the total mass and the symmetric mass ratio of the system, respectively, $\delta M = m_1 - m_2$, i is the inclination of the plane of the binary with respect to the line-of-sight and φ is the orbital phase whose post-Newtonian expansion is also known presently to order v^5. For equal mass binaries, that is, $\delta M = 0$ and $\eta = 1/4$, the PN amplitude corrections $H^{(n)}_{+,\times}$, $n > 0$, are less important than they are for asymmetric binaries with $\delta M \sim M$ and $\eta \ll 1$. Although these *amplitude* corrections can be significant in certain stellar mass binaries, which the initial ground-based interferometers are likely to observe, such as a NS-BH binary, it has been customary to employ only the dominant term $H^{(0)}_{+,\times}$ in search templates – the so-called *restricted* PN approximation [9]. We shall continue to adopt this approximation in this paper, but let us note here that a careful search for asymmetric binaries must include the full PN signal including all the amplitude corrections [10]. In the restricted PN approximation the signal recorded by the detector is the linear combination

$$h = F_+ h_+ + F_\times h_\times, \tag{3}$$

where F_+ and $F_\times$ denote the antenna patterns of the detector [11].

The evolution of the orbital phase can be worked out by using the PN expansions of the binding energy per unit mass $E(v)$ of the system and the gravitational wave flux $F(v)$ emitted, in an energy balance equation, namely, $F = -M(dE/dt)$. For two bodies of comparable masses in circular orbit one has

$$E(v) = -\frac{\eta v^2}{2}\left(1 + E_2 v^2 + E_4 v^4 + O(v^6)\right), \tag{4}$$

$$F(v) = \frac{32\eta^2 v^{10}}{5}\left(1 + F_2 v^2 + F_3 v^3 + F_4 v^4 + F_5 v^5 + O(v^6)\right), \tag{5}$$

where

$$E_2 = -\frac{9+\eta}{12}, \quad E_4 = -\frac{81 - 57\eta + \eta^2}{24}, \tag{6}$$

$$F_2 = -\frac{1247}{336} - \frac{35\eta}{12}, \quad F_3 = 4\pi, \quad F_4 = -\frac{44711}{9072} + \frac{9271\eta}{504} + \frac{65\eta^2}{18}, \quad F_5 = -\left(\frac{8191}{672} + \frac{535\eta}{24}\right)\pi. \tag{7}$$

The phasing of the orbit is given parametrically by the following pair of ordinary differential equations (ODEs):

$$\frac{d\varphi}{dt} = \frac{v^3}{M}, \qquad \frac{dv}{dt} = \frac{dv}{dE}\frac{dE}{dt} = -\frac{F(v)}{ME'(v)} \tag{8}$$

where $E'(v) \equiv dE/dv$. There are several different, but conceptually equivalent, possibilities to proceed from the above equations in arriving at an explicit phasing formula. A straightforward approach is to simply substitute for E and F their PN expansions, Eqs. (4) and (5), re-expand the rational polynomial F/E' to the correct PN order and solve the above differential equations to yield an explicit time-domain phasing formula. This is the usual PN approximation that has also been called the T-approximant [2]. Alternatively, given the PN expansions, or other representations, of the flux and energy, one can numerically solve the above ODEs for equal time steps and then use Eq. (3). It turns out that this latter method is accurate and computationally quite fast. Other equivalent, but numerically distinct, approaches are discussed in Ref. [1], to which we refer the interested reader for an exhaustive account and comparison. Among the different approaches introduced therein, we would like to discuss here one in some detail, viz the P-approximants [2, 3]. P-approximants make the best use of our current theoretical knowledge and combine that with a fast converging re-summation technique. We shall compare its performance against a recently proposed new class of waveforms, based on a different re-summation technique [5] – the effective one-body waveforms – which is likely to serve as the most accurate set of search templates.

3 Effective One-Body Approach

As discussed above, in the standard "adiabatic approximation" the evolution of the orbital phase is constructed by combining the energy-balance equation $M(dE/dt) = -F$ with either the PN approximation, or re-summed estimates, for the energy and flux as functions of the instantaneous circular orbital frequency. Recently, Buonnano and Damour [5] introduced a new approach which is no longer limited to the adiabatic approximation and expected to describe rather accurately the transition between the inspiral and the plunge, and to give also an estimate of the ensuing plunge signal. The approach of [5] is essentially, like [2, 3], a re-summation technique which consists of two main ingredients: (i) the "conservative", that is non-dissipative, part of the dynamics [effectively equivalent to the specification of the $E(v)$ in the previous approaches] is re-summed by a an effective one-body dynamics that replaces the two-body dynamics, and (ii) the dissipative part of the dynamics [equivalent to the specification of the $F(v)$] is constructed by borrowing the re-summation technique introduced in [2]. In practical terms, the time-domain signal at the detector is given by the following expression in terms of the reduced time $\hat{t} = t/M$:

$$h(\hat{t}) = \mathcal{C}\, v_\omega^2(\hat{t}) \cos(2\varphi(\hat{t}))\,, \quad v_\omega \equiv \left(\frac{d\varphi}{d\hat{t}}\right)^{\frac{1}{3}}\,, \tag{9}$$

where $\mathcal{C}$ is a constant for a given binary depending on the masses of the two stars, the polarisation of the wave, the antenna pattern and the distance to the source. The orbital phase $\varphi(\hat{t})$ entering Eq. (9) is given by integrating a system of ODE's:

$$\frac{dr}{d\hat{t}} = \frac{\partial \widehat{H}}{\partial p_r}(r, p_r, p_\varphi)\,, \tag{10}$$

$$\frac{d\varphi}{d\hat{t}} = \widehat{\omega} \equiv \frac{\partial \widehat{H}}{\partial p_\varphi}(r, p_r, p_\varphi)\,, \tag{11}$$

$$\frac{dp_r}{d\hat{t}} + \frac{\partial \widehat{H}}{\partial r}(r, p_r, p_\varphi) = 0\,, \tag{12}$$

$$\frac{dp_\varphi}{d\hat{t}} = \widehat{F}_\varphi(\widehat{\omega}(r, p_r, p_\varphi))\,. \tag{13}$$

The reduced Hamiltonian $\widehat{H}$ (of the associated one-body problem) is given, at the 2PN approximation[1], by

$$\widehat{H}(r, p_r, p_\varphi) = \frac{1}{\eta}\sqrt{1 + 2\eta\left[\sqrt{A(r)\left(1 + \frac{p_r^2}{B(r)} + \frac{p_\varphi^2}{r^2}\right)} - 1\right]}\,, \tag{14}$$

$$\text{where} \;\; A(r) \equiv 1 - \frac{2}{r} + \frac{2\eta}{r^3}\,, \qquad B(r) \equiv \frac{1}{A(r)}\left(1 - \frac{6\eta}{r^2}\right). \tag{15}$$

The damping force F_φ is approximated by

$$\widehat{F}_\varphi = -\frac{1}{\eta v_\omega^3} F_{P_n}(v_\omega)\,, \tag{16}$$

where $F_{P_n}(v_\omega) = \frac{32}{5}\,\eta^2\, v_\omega^{10}\, \hat{F}_{P_n}(v_\omega)$ is the flux function used in P-approximants to be discussed below.

The system of equations (10)–(13), allows one to describe the smooth transition which takes place between the inspiral and the plunge, while the adiabatic evolution becomes spuriously singular at the LSO, where $E'(v_{\mathrm{LSO}}) = 0$. Ref.[5] advocated to continue using the above system of equations after the transition, to describe the waveform emitted during the plunge and to match around the "light ring" to a "merger" waveform, described, in the first approximation, by the ringing of the least-damped quasi-normal mode of a Kerr black hole [5]. At the moment this technique is the most complete. It includes, in the best available approximation and for non-spinning black holes, most of the correct physics of the problem, and leads to a specific prediction for the complete waveform (inspiral + plunge + merger)

[1] The 3PN version of $\widehat{H}$ has been recently obtained in Ref. [4].

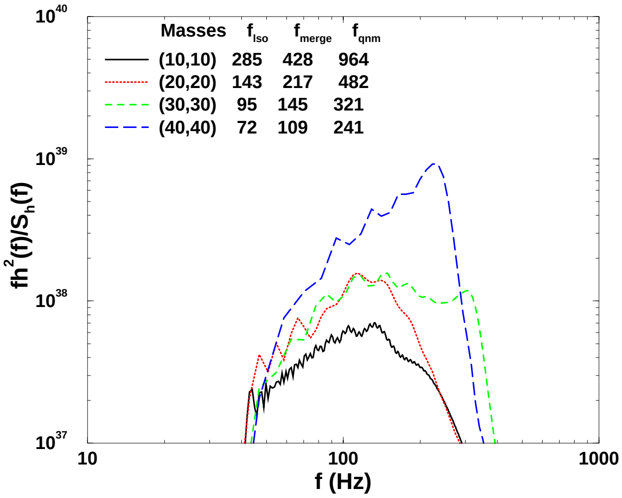

Figure 1: Power spectral density of the signal weighted down by the noise spectral density for effective one-body waveforms for different binary sources.

emitted by coalescing binaries. Because of its completeness, we shall use it as our "fiducial exact" waveform in our comparison between different search templates.

The initial data $(r_0, p_r^0, p_\varphi^0)$ are found using

$$r_0^3\left[\frac{1+2\eta(\sqrt{z(r_0)}-1)}{1-3\eta/r_0^2}\right]-\hat{\omega}_0^{-2}=0,\quad p_\varphi^0=\left[\frac{r_0^2-3\eta}{r_0^3-3r_0^2+5\eta}\right]^{1/2},\quad p_r^0=\frac{F_\varphi(\hat{\omega})}{C(r_0,p_\varphi^0)(dp_\varphi^0/dr_0)} \tag{17}$$

where $z(r)$ and $C(r, p_\varphi)$ are given by

$$z(r)=\frac{r^3A^2(r)}{r^3-3r^2+5\eta},\quad C(r,p_\varphi)=\frac{1}{\eta\widehat{H}(r,0,p_\varphi)\sqrt{z(r)}}\frac{A^2(r)}{(1-6\eta/r^2)}. \tag{18}$$

The plunge waveform is terminated when the radial coordinate attains the value at the light ring $r_{\rm lr}$ given by the solution to the equation,

$$r_{\rm lr}^3-3r_{\rm lr}^2+5\eta=0. \tag{19}$$

The subsequent "merger" waveform is constructed as in Ref.[5].

In Fig. 1 we show the power spectrum of the effective one-body waveform weighted by the noise power spectral density of LIGO for four different systems. For binaries of mass less than about 30 $M_\odot$ the signal exhibits mostly the inspiral part, for $30M_\odot < M < 50M_\odot$ we see the inspiral and plunge while for $M > 50M_\odot$ one sees the plunge and the quasi-normal mode ringing, the inspiral part being insignificant.

4 P-approximants

The PN series for the flux and energy functions, Eqs. (4) and (5), are asymptotic series that converge only in the limit $v \to 0$. In our application, however, we will be using these

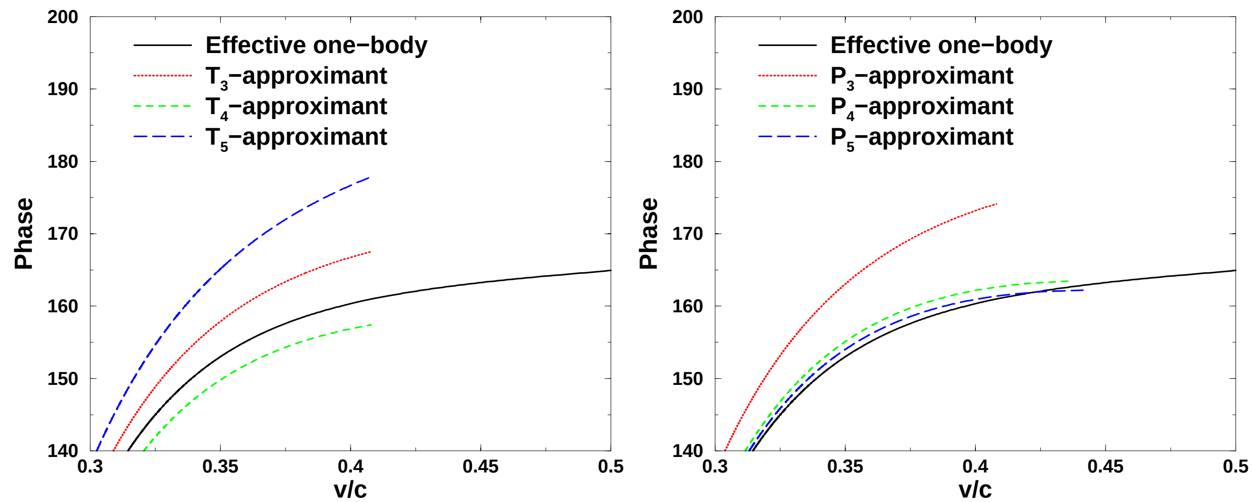

Figure 2: Orbital evolution of a binary consisting of two stars of equal masses is shown plotted at different post-Newtonian orders. We show the usual Taylor approximant-based phase evolution on the left and the re-summed P-approximant-based phase evolution on the right.

series in a region where $v \sim 1$. It is no surprise, therefore, that one finds that the orbital evolution inferred by solving the ODEs in Eq. (8) differ greatly at different PN orders. More precisely, the orbital evolution inferred by using the post-Newtonian expansions of E and F in Eq. (8) are significantly different even at high PN orders when v is close to 1 or when the binary system is close to coalescence. This is shown in Fig. 2 where we have plotted $\varphi(v)$ as a function of v at different PN orders. Also plotted in thick solid line is the orbital evolution predicted by the effective one-body approach discussed in the previous Section, which is, presently, our best guess on how the phase may evolve. In the limiting case of a test mass orbiting a Schwarzschild black hole the post-Newtonian expansion of the phase can be worked out up to order $O(v^{11})$. Even at such high PN orders the expansion is poorly convergent [12, 2]. This poor convergence means that the PN expansions of the phase are not very reliable and their use in constructing templates will result in missing up to one in every three events that a detector can observe with the aid of a more accurate template [2]. More importantly, the measurement of physical parameters will be severely biased, in some cases by as much as 100 %, thereby making gravitational astronomy imprecise.

The situation can be dramatically improved by using re-summation techniques, which work with quantities that are conceptually equivalent to the PN expansions, but numerically different from them. These re-summed quantities are often rapidly convergent because they can capture a certain physical effect which a PN expansion is never able to. For instance, on very general grounds it is expected that the PN expansion of the flux of GW emitted by a test particle inspiralling in Schwarzschild geometry must exhibit a simple-pole singularity at the location of the light-right [2], namely $v = 1/\sqrt{3}$. A straightforward PN expansion of the flux will never be able to predict this pole while a rational-polynomial approximation to the post-Newtonian series, will by construction, yield a pole.

It is often the case in physical problems, that such rational-polynomials, called Padé approximants, capture the location of the true pole singularity. However, to be able to employ this technique one must work with the correct physical quantities so that, to the best of available knowledge, only a pole singularity occurs in this quantity, and not some other form of singularity, such as a branch point. To apply the Padé techniques successfully, therefore, we have had to begin from more 'basic' quantities. For instance, instead of working with the usually defined energy function $E(v)$, which is, incidentally, asymmetric in the two masses, we defined the energy function $e(v)$

$$e(v) = \left[1 + \frac{1}{2\eta}\left(E^2 + 2E\right)\right]^2 - 1, \tag{20}$$

which we know in the test mass limit possesses only a simple pole on the real line and is also symmetric in the two masses. Indeed, as shown in Ref. [2], the rational polynomial constructed from just the first two terms in the PN expansion[2] $e_{T_2}(v)$ reproduces the exact energy $e(v)$. Hence, by using, in the case of binaries with two comparable masses, this very form energy function as the quantity on which to apply the Padé technique, we are more likely to produce the correct pole singularity. As in the case of energy, we use a new flux function as a starting point and use the Padé approximant of the new energy and flux functions in constructing the phase evolution. We call this approach of introducing new energy and flux functions and then applying re-summation techniques as the P-approximant approach.

The P-approximant phase evolution is plotted on the right hand panel of Fig. 2 and again compared to the effective one-body approach. Clearly, the curves on the right panel are more rapidly convergent and more closer to the effective one-body phase evolution, than the Taylor approximant phase evolution on the left panel.

5 Overlaps

The visual comparison of the phase evolution discussed in the previous Section is a qualitative test of the accuracy of different approximation schemes. Eventually, our interest lies in the construction of template waveforms that are effectual in capturing the true signal. Therefore, of primary concern is the fraction of the optimal signal-to-noise ratio (SNR)[3] captured by an approximate waveform. In maximising this fraction one varies both the extrinsic, that is the initial phase and the time-of-arrival, as well as the intrinsic, that is the masses of the component stars, template parameters. The SNR depends not only on the properties of the signal but also on the frequency response of the detector. More precisely, the fraction of the optimal SNR achievable by a given approximant $\mathcal{O}$ is given by

$$\mathcal{O} = \max_{\text{parameters}} \frac{\langle A, X\rangle}{\sqrt{\langle A, A\rangle \langle X, X\rangle}}, \tag{21}$$

where, given two functions $A(t)$ and $B(t)$ their scalar product $\langle A, B\rangle$ is defined as

$$\langle A, B\rangle = 2\int_0^\infty \frac{df}{S_h(f)} \tilde{A}(f)\tilde{B}^*(f) + \text{C.C.} \tag{22}$$

[2] We follow the convention wherein the Taylor expansion of a quantity $f(v)$ to order v^n is denoted as $f_{T_n}(v)$. Thus, $f(v) = f_{T_n}(v) + O(v^{n+1})$.

[3] Optimal SNR is that SNR which would have been achieved had we employed the functional form of the true signal as a template in matched filtering the detector output.

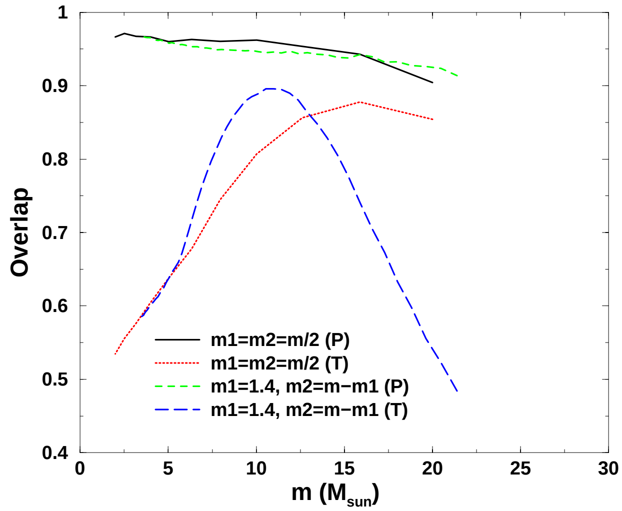

Figure 3: The overlap of T- and P-approximant waveforms with effective one-body waveforms.

Here $\tilde{A}(f)$ denotes the Fourier transform of $A(t)$ and $\tilde{A}^*(f)$ is the complex conjugate of $\tilde{A}(f)$ and $S_h(f)$ denotes the one-sided noise power spectral density of the detector.

In Fig. 3 we have plotted the overlap of T- and P-approximants with the effective one-body waveforms as a function of the total mass, for equal mass binaries, and for a system in which one of the bodies is a neutron star. We have maximised the inner product only over extrinsic parameters in order to exhibit how faithful are the two approximants in reproducing the effective one-body waveform. The superiority of P-approximants over T-approximants is quite clearly brought about in this plot. For binaries whose merger takes place at frequencies beyond the sensitive bandwidth of a detector the disagreement between P-approximants and the effective one-body approach is not too great. However, for sources that merge in the frequency band of the detector, it is best to employ effective one-body waveforms as search templates.

Having demonstrated that the two re-summed techniques give very similar waveforms we suggest that that the effective one-body waveforms be used as search templates. Indeed, a good strategy would be to employ a set of search templates based on, but encompassing a larger space than, the waveforms derived from the effective one-body approach.

References

[1] T. Damour, B.R. Iyer and B.S. Sathyaprakash, *A comparison of search templates for gravitational waves from binary inspiral,* Phys. Rev. D (in press) gr-qc/0010009 (2000).

[2] T. Damour, B.R. Iyer, and B.S. Sathyaprakash, Phys. Rev. D **57**, 885 (1998).

[3] T. Damour, B.R. Iyer, and B.S. Sathyaprakash, Phys. Rev. D **62**, 084036 (2000).

[4] T. Damour, P. Jaranowski and G. Schäfer, Phys. Rev. D **62**, 021501; *ibid,* **62**, 044024; *ibid,* **62**, 084011 (2000).

[5] A. Buonanno and T. Damour, Phys. Rev. D **62,** 064015 (2000).

[6] V.M. Lipunov, K.A. Postnov, and M.E. prokhorov, New Astron. **2,** 43 (1997);

[7] L. Blanchet, T. Damour, B.R. Iyer, C.M. Will and A. Wiseman, Phys. Rev. Lett. **74**, 3515 (1995); L. Blanchet, Phys. Rev. D **54**, 1417 (1996).

[8] L. Blanchet, B.R. Iyer, C.M. Will and A. Wiseman, Class. and Quantum Grav. **13,** 575 (1996).

[9] C. Cutler, et. al. Phys. Rev. Lett. **70**, 2984 (1993).

[10] A. Sintes and A. Vecchio, *Detection of gravitational waves from inspiralling compact binaries using non-restricted post-Newtonian approximations,* gr-qc/0005058.

[11] K.S. Thorne, in, *Three Hundred Years of Gravitation,* Eds. S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987) pp 330.

[12] E. Poisson, Phys. Rev. D **52**, 5719 (1995).